The first English translation of selections from Bernoulli's groundbreaking work identifying the curve formed by a hanging chain suspended at both ends.

Johann Bernoulli (1667-1748)

(Translated by William A. Ferguson, Jr.)

The editor's note reads:

In 1690, Jakob Bernoulli, brother of Johann, published a challenge to the scientific world in the Acta Eruditorum of Leipzig, to determine the geometry of the catenary. Johann Bernoulli, Gottfried Leibniz, and Christiaan Huygens each independently solved the problem. All three solutions were published in the Acta in 1691.

Johann Bernoulli then treated the subject in his Lectures on the Integral Calculus, which were written out for the use of the mathematician Guillaume Marquis de L'Hôpital in 1691-1692, while Bernoulli was residing in Paris. The excerpts presented here are from this work.

Part I is on the physics of the hanging chain. Part II provides the derivation of the differential equation of the curve whose geometry corresponds to the physics of the hanging chain. Part III is the proof that Leibniz's solution, based on the logarithmic curve, is identical. These three parts are Lecture Thirty-Six, part of Lecture Twelve, and Lecture Thirty-Seven, respectively.

The text is from Die erste Integralrechnung, Eine Auswahl aus Johann Bernoullis mathematischen Vorlesungen über die Methode der Integrale (Leipzig and Berlin: Wilhelm Engelmann, 1914), itself a translation into German by Gerhard Kowalewski of a selection of lectures from the Latin original, Lectiones mathematicae de methodo integralium. The figures are reproduced from the Kowalewski translation. The reader should not assume that the figures are exact constructions. The Kowalewski translation may be found at http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=Bern002&seq=5

All material in square brackets has been supplied by the translator. The footnotes are by the German (Kowalewski) or the English translator (Ferguson), as noted in square brackets at the end of each note.

The translator thanks the staff of the Burndy Library, of the Dibner Institute for the History of Science and Technology, Massachusetts Institute of Technology, Cambridge, Massachusetts, for making available a copy of the Latin original, Lectiones mathematicae de metodo integralium, as an aid to translation.
1. Lecture Thirty-Six
On Catenaries

The importance of the problem of the catenary in Geometry can be seen from the three solutions in the Acta of Leipzig of last year (1691), and especially from the remarks that the renowned Leibniz makes there. The first to consider this curve, which is formed by a free-hanging string, or better, by a thin inelastic chain, was Galileo. He, however, did not fathom its nature; on the contrary, he asserted that it is a parabola, which it certainly is not. Joachim Jungius discovered that it is not a parabola, as Leibniz remarked, through calculation and his many experiments. However, he did not indicate the correct curve for the catenary. The solution to this important problem therefore remained for our time. We present it here together with the calculation, which was not appended to the solution in the Acta.

There are actually two kinds of catenaries: the common, which is formed by a string or a chain of uniform thickness, or is of uniform weight at all points, and the uncommon, which is formed by a string of non-uniform thickness, which therefore is not of uniform weight at all points, and certainly not uniform in relation to the ordinates of any given curve.

Before we set about the solution, we make the following assumptions, which can easily be proven from Statics.

1. The string, rope, or chain, or whatever the curve consists of, will be assumed to be flexible and inelastic at all of its points, that is, it undergoes no stretching as a result of its weight.

2. If the catenary $ABC$ [Figure 1] is held fixed at any two points $A$ and $C$, then the necessary forces at points $A$ and $C$, are the same as those which support a weight $D$, that is equal to the weight of the chain $ABC$ and is located at the meeting point of two weightless strings $AD$ and $CD$, that are tangent to the curve at points $A$ and $C$. The reason for this is clear: Because the weight of the chain $ABC$ exerts its action at $A$ and $C$ in one direction [at each point], namely, in the directions of the tangents $AD$ and $CD$, and the pull of the same or equal weight $D$ at $A$ and $C$ likewise goes in the directions of $AD$ and $CD$. Therefore the necessary forces at points $A$ and $C$ must also in both cases be the same. Accordingly, one obtains the necessary force at the lowest point $B$, when one seeks the force that the weight $E$ [Figure 2] exerts at the same point, when it is held by two weightless strings, one of which is tangent to the curve at $B$, and therefore is horizontal, while the other is tangent to the curve at point $A$.

3. When a chain fastened at points $A$ and $C$ is then fastened at any other point $F$ [Figure 3], so that one could remove the portion $AF$, the curve represented by the remaining piece of chain $FBC$ does not change, that is, the remaining points will stay in the same position as before the fastening [at $F$]. This needs no proof, because Reason advises it and experience lays it daily before our eyes.

4. If we retain the previous assumptions, then before and after the fastening [at $F$], the same (that is, the original) force must obtain at particular positions on the curve, or, what amounts to the same thing, a point will be pulled with the same force after the fastening [at $F$] as before it. This is nothing but a corollary of the preceding number. Consequently, as one lengthens or shortens the chain $BFA$, that is, wherever one chooses the fastening point $F$, the force at the lowest position $B$ neither increases nor decreases, but always remains the same.

5. The weight $P$ [Figure 4], which is held by any two arbitrarily situated strings $AB$ and $CB$, exerts its forces on the points $A$ and $C$ in such a relation, that the necessary force at $A$ is to the necessary force at $C$ (after drawing vertical line $BG$), as the sine of angle $CBG$ is to the sine of angle $ABC$, and the force of the weight $P$ is to the force at $C$ as the sine of the whole angle $ABC$ is to the sine of the opposite angle $ABG$. This is proven in every theory of Statics.

With these assumptions, we find the common catenary curve in the following manner. Let $BAa$ be the desired curve [Figure 5]; $B$, its deepest point; the axis or the vertical through $B$, $BG$; the tangent at the deepest point, which will be horizontal, $BE$; and let $AE$ be the tangent at any other point $A$. 

\[ \text{Figure 1} \]

\[ \text{Figure 2} \]

\[ \text{Figure 3} \]
Draw the ordinate $AG$ and the parallel $EL$ to the axis. Let

![Figure 4]

and the weight of the chain, or, since it is of uniform thickness, the length of the curve $BA = s$. Since at point $B$, an ever constant force will be required (by assumption 4), whether the chain be lengthened or shortened, that force, or the segment $C = a$ expressing it, will therefore be a constant. Imagine now that the weight of the chain $AB$ is concentrated at and hangs at the meeting point $E$ of the tangent strings $AE, BE$; then (by assumption 2) the same force is required at point $B$ to hold the weight $E$ as was required to hold the chain $BA$. But the weight $E$ (by assumption 5) is to the force at $B$, as the sine of the angle $AEB$, or as the sine of its complementary angle $EAL$ is to the sine of angle $AEL$, that is, as $EL$ is to $AL$. Wherever on the curve one chooses the fixed point $A$ (the curve always remains the same, by assumption 3), the weight of the chain $AB$ is to the force at $B$ [which force equals the constant $a$], as $EL$ is to $AL$, that is,

$$s : a = EL : AL = AH : Ha = dx : dy$$

and if one inverts,

$$dy : dx = a : s.$$  

Hence it follows that the catenary $BA$ is the same as that curve whose construction and nature we have given above, by the method of inverse tangents [provided here as Part II, below], where we first converted the proportion $dy : dx = a : s$ to the following:

$$dy = \frac{adx}{\sqrt{2adx + x^3}}$$

at which point the curve was constructed through the rectification of the parabola as well as through the quadrature of the hyperbola.

II. Part of Lecture Twelve on Inverse Tangents

IV. To find the nature of the curve so created that $DC : BC = E : AD$ [Figure 6].

![Figure 6]

Let $AC = x$, $CD = y$, $AD = s$ [and the given constant segment $E = a$]. By assumption,

$$\frac{dy}{dx} = \frac{a}{s}, \text{ therefore } dy = \frac{adx}{s}.$$  

However, to be able to eliminate the letter $s$ (which is always necessary in the determination of curves), one must proceed thus:

$$dy^2 = \frac{a^2dx^2}{s^2},$$

therefore,

$$ds^2 = dx^2 + dy^2 = \frac{s^2dx^2 + a^2dx^2}{s^2}$$

and

$$ds = \frac{dx\sqrt{s^2 + a^2}}{s},$$

therefore

$$dx = \frac{sds}{\sqrt{s^2 + a^2}}$$

and the integral thereof,

$$x = \sqrt{s^2 + a^2}.$$  

From this is obtained

$$s = \sqrt{x^2 - a^2}.$$
and

\[ ds = \frac{adx}{\sqrt{x^2 - a^2}} = \sqrt{dx^2 + dy^2}. \]

If the equation is simplified, one obtains

\[ x^2 dy^2 - a^2 dy^2 = a^2 dx^2 \]

and finally

\[ dy = \frac{adx}{\sqrt{x^2 - a^2}}. \]

The same result is achieved otherwise and more easily in the following manner.

Because \( s = adx / dy \), then \(^5\)

\[ ds = \sqrt{dx^2 + dy^2} = \frac{ad^2 x}{dy} \]

and hence

\[ dy = \frac{ad^2 x}{\sqrt{dx^2 + dy^2}}. \]

To be able to take the integral on both sides, both sides are multiplied by \( dx \). Then one obtains

\[ dx dy = \frac{adx dx^2}{\sqrt{dx^2 + dy^2}}. \]

If the integral be taken, the result is

\[ x dy = a \sqrt{dx^2 + dy^2}, \]

and after simplifying the equation,

\[ dy = \frac{adx}{\sqrt{x^2 - a^2}}, \]

as before.

Now we come to the construction of this curve. It should be noted, first of all, that because \( x = \sqrt{s^2 + a^2} \) and hence \( x > s \), the invariant origin of \( x \) lies beyond the vertex \( B \), and indeed at the distance \( E \), since if \( s = 0 \), then \( x = a \) necessarily. Hence if we wish to place the origin of \( x \) at the vertex itself, we must set \( x = x + a \). Then the equation \( dy = adx / \sqrt{x^2 - a^2} \) is transformed into the following:

\[ dy = \frac{adx}{\sqrt{2ax + x^2}}, \]

which we will now construct in a three-fold way.

Multiply the equation by \( a \). Then

\[ ady = \frac{a^2 dx}{\sqrt{2ax + x^2}}. \]

Now draw the normals \( AK, GH \), which will intersect at \( B \) [Figure 7]; take \( BA = a \) and draw with vertex \( B \) and midpoint \( A \) the equilateral hyperbola \( BC \). Construct further a curve \( DJ \), such that \( BA \) is everywhere the mean proportional between \( KC \) and \( KD \), that is, that

\[ KD = \frac{a^2}{\sqrt{2ax + x^2}}. \]

Next, draw the parallel \( AF \) and make the area of the rectangle \( AG \) equal to that of \( HBkDJ \). If we now extend \( DK \) and \( FG \), their intersection \( E \) will lie on the desired curve.\(^6\)

The curve admits of another and easier construction in the following manner. Draw the line \( AC \) and make the area of rectangle \( AG \) equal to double the hyperbolic area \( ABC \). Then after the extension of \( CK \) and \( FG \), the point \( E \) again will lie on the same desired curve.\(^7, 8\)

It may be constructed in yet another way, by means of the rectification of a parabolic curve, in the following manner.

Because \( dy = adx / \sqrt{2ax + x^2} \), therefore

\[ dy + \frac{adx + dx}{\sqrt{2ax + x^2}} (\text{differential of } EK + KC = EC) = \frac{2adx + dx}{\sqrt{2ax + x^2}} = \frac{dx \sqrt{2a + x}}{\sqrt{x}}. \]

Hence one must find a certain curve \( BL \), whose differential is

\[ dx \sqrt{2a + x} / \sqrt{x}. \]

Then \( BL \) itself will be equal to \( EC \). But one finds this curve thus: From

\[ \frac{2adx^2 + x dx^2}{x}, \]

subtract \( dx^2 \). Then \( 2adx^2/x \) remains. Hence

\[ \frac{dx \sqrt{2a}}{\sqrt{x}} = \text{differential of } KL, \]

and

\[ \int \frac{dx \sqrt{2a}}{\sqrt{x}}, \text{ that is, } \sqrt{8ax}, \]
will therefore be equal to $KL$ itself. The curve $BL$ is consequently a parabola of parameter $8AB$. If this $BL$ be laid out as a straight line and abutted to point $C$ as ordinate, the other endpoint $E$ again will lie on the desired curve $BE$.9,10

**Corollary.** The length of the curve $BE$ is equal to the hyperbolic ordinate $KC$. If the length of $BE$ is designated by $s$, then

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + \frac{a^2dx^2}{2ax + x^2}} = dx \sqrt{\frac{a^2 + 2ax + x^2}{2ax + x^2}} = \frac{adx + xdx}{\sqrt{2ax + x^2}}.$$ 

Therefore $PQ = a\sqrt{2}$ and $\frac{1}{2}DM + \frac{1}{2}PQ$, that is, $DA$ by construction,

$$z = \frac{a^2 + x^2}{2},$$

**Therefore**

$$z^2 = 2ax + 2x^2 - a^2.$$ 

When solved, this equation yields,

$$z = a + x + \sqrt{2ax + x^2}.$$ 

Therefore

$$dz = dx + \frac{(a + x)dx}{\sqrt{2ax + x^2}}.$$ 

If one substitutes this value of $z$ in the earlier equation $dz = zdy/a$, the result is

$$dx + \frac{(a + x)dx}{\sqrt{2ax + x^2}} = \frac{ady + xdy + dy \cdot \sqrt{2ax + x^2}}{a}$$

or

$$\frac{adx \cdot \sqrt{2ax + x^2} + a^2dx + axdx}{\sqrt{2ax + x^2}} = ady + xdy + dy \cdot \sqrt{2ax + x^2}.$$ 

Dividing both sides by $a + x + \sqrt{2ax + x^2}$, one obtains

$$\frac{adx}{\sqrt{2ax + x^2}} = dy.$$ 

Because this equation is the same as the one that we have found, it follows that the curve $BA$ is also our catenary, and that the Leibnizian construction, as different as it may be from the one we have given above, indeed produces no other line.

It remains to add the most important properties of the simple catenary, and that with calculation and proof, which was not done in the Acta. We will use the figure that appears in the Acta [Figure 9]. There, $EBF$ is the catenary, $B$ its deepest point, $BA$ the axis, $BG$ the equilateral hyperbola that can be termed generative, and $BH$ the parabola, through whose rectification the catenary line $EBF$ is constructed.
1. Draw the tangent $FD$; then $AF : AD = BC : BF$, the curve, because 

$$AF : AD = dy : dx.$$ 

However we found by calculation that 

$$dy : dx = a : s.$$ 

Therefore, the statement is correct.

2. $AE$ or $AF$ is equal to the parabolic curve $BH$, minus the segment $AG$. This is clear, because by construction, $EG$ was taken equal to $BH$.

3. The length of the curve $BE$ or $BF$ is equal to the segment $AG$; that is, the portions of the catenary, if one lays them out upon the axis as ordinates, form an equilateral hyperbola. That is a remarkable property of this curve. We have proven this by the method of inverse tangents (see page 31).

4. The area of the catenary region $BAE$ or $BAF$ is equal to the segment $AG$; that is, the portions of the catenary, if one lays them out upon the axis as ordinates, form an equilateral hyperbola. Since of course $\therefore$ therefore $a$ or $CB$ is to $a + x$ or $CA$ as $CA$ is to $EO$.

5. The length of the curve $MNO$, the involute of which forms the catenary $BE$, is the third proportional of $CB$ and $AG$. To discover that this is so, one would first find the length of the unwinding tangent line $EO$; above, in the article on the development of curves [in Lecture Sixteen, not translated here], we have shown that in general, for all curves, it is equal to

$$\frac{(dx^2 + dy^2) \sqrt{dx^2 + dy^2}}{-d^3ydx}.$$ 

For the curve in question then, 

$$dx^2 + dy^2 = ds^2 = \frac{(a^2 + 2ax + x^2) dx^2}{2ax + x^2},$$ 

and, since 

$$dy = \frac{adx}{\sqrt{2ax + x^2}},$$ 

hence 

$$d^3y = -\frac{(a^2 + ax)dx^2}{(2ax + x^2)\sqrt{2ax + x^2}}.$$ 

From this is obtained, for the whole expression, 

$$\frac{(dx^2 + dy^2) \sqrt{dx^2 + dy^2}}{-d^3ydx} = EO = \frac{a^2 + 2ax + x^2}{a}.$$ 

From this, subtract that which the assumption $x = 0$ yields; then what remains is 

$$\frac{2ax + x^2}{a} = \text{the length of curve MNO}.$$ 

Therefore $a$ or $CB$ is to $\sqrt{2ax + x^2}$ or $AG$, as $AG$ is to $MNO$.

6. The unwinding tangent $EO$ is the third proportional of $CB$ and $CA$. For, since 

$$EO = \frac{a^2 + 2ax + x^2}{a},$$ 

$a$ or $CB$ is to $a + x$ or $CA$ as $CA$ is to $EO$.

7. The line $BM$, which extends to the beginning of the curve $MNO$, is equal to $CB$. Since of course $x = 0$, the unwinding tangent $EO$, which is now $BM$, becomes equal to $a = CB$.

8. $MP$ is twice $BA$. Because 

$$MNO = \frac{2ax + x^2}{a},$$ 

the differential will be 

$$Oo = \frac{(2a + 2x)dx}{a}.$$ 

But the triangle $Oop$ is similar to triangle $ESR$ and hence also to triangle $eER$; consequently $Ee : ER = Oo : po$, that is, 

$$\frac{adx + xdx}{\sqrt{2ax + x^2}} : \frac{adx}{\sqrt{2ax + x^2}} = \frac{2adx + 2xdx}{a} : po,$$
therefore $2dx = po$ and, if integrated, $2x = PM$.  

9. The area of the rectangle of $CB$ and $PO$ is twice that of the hyperbolic area $ABG$. Since of course $Ee : Oo = eR : pO$, that is,

$$\frac{adx + edx}{\sqrt{2ax + x^2}} : \frac{2adx + 2edx}{a} = dx : pO,$$

therefore

$$pO = \frac{2adx - \sqrt{2ax + x^2}}{a},$$

thus

$$\frac{CB \cdot pO}{a} = 2dx \sqrt{2ax + x^2},$$

and, when integrated,

$$\frac{CB \cdot PO}{a} = \text{double the hyperbolic area } ABG.$$ 

10. The segment $CP$ is bisected by the point $A$. Since of course $MP = 2x$, the result is that $BP = 2x + a$, $BP - BA$ or $AP = x + a = CA.$

11. The length of the curve $EB$ is to the length of the curve $MNO$ as the segment $CB$ is to segment $AG$. That is because $EB$, that is, $\sqrt{2ax + x^2}$, is to $MNO$ or

$$\frac{2ax + x^2}{a}$$

as a is to $\sqrt{2ax + x^2}$, and therefore as $CB$ is to $AG$.

12. If the two rectangles $AJ$, $AK$ are placed on $AG$ ($AJ$ is formed from the half transverse axis $CB$ and the segment $FG$, and the area of $AK$ is equal to that of the hyperbolic area $BGA$); and, from the vertex $B$ along the axis, a segment $BL$ equal to width $KJ$ is drawn; then point $L$ will be the center of gravity of the catenary $EBF$. This will be proven in another location.

13. If one imagines infinitely many curves drawn from $E$ to $F$ that are equal in length to the catenary $EBF$ and lays them out as straight lines, and constructs individual vertical lengths at the individual points of the particular segment equal to the respective distances from the line $EF$, then of all areas formed in this manner, the greatest is that formed by the catenary.

This will be proven with the help of the axiom that the center of gravity descends as far as it can.

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Two huge chain catenaries used in the testing of aircraft at NASA’s Vertical Motion Flight Simulation Laboratory.
1. In Galileo’s *Dialogues Concerning Two New Sciences* (1638), the catenary is discussed and its form compared to a parabola. Joachim Jungius (1669) showed that the curve is not a parabola, but he was not really able to define it. Jakob Bernoulli proposed the problem anew in May 1690 in the *Acta Eruditorum,* and then it was solved by Huygens, Leibniz, and Johann Bernoulli. [Kowalewski]

2. Let the force acting along $AB$ be $F_A,$ the force acting along $CB$ be $F_C,$ and the weight of $P$ be $F_p.$ The horizontal component of $F_A$ is $F_A \sin \angle ABG,$ opposing the horizontal component of $F_C,$ which is $F_C \sin \angle CBG;$ and $F_p$ is the force of gravity on $P,$ and only acts vertically. Because $P$ is not moving horizontally, or

$$F_p = F_A \cos \angle ABG - F_C \cos \angle CBG = 0,$$

or

$$F_A : F_C = \sin \angle ABG \cdot \sin \angle CBG,$$

as stated in point 5 of this lecture. The vertical component of $F_A$ is $F_A \cos \angle ABG,$ pulling upward; the vertical component of $F_C$ is $F_C \cos \angle CBG,$ also pulling upward; and $F_p,$ the weight of $P,$ is of course pulling down. Because $P$ is not moving vertically, we have

$$F_p = F_A \cos \angle ABG + F_C \cos \angle CBG,$$

Dividing both sides of the equation by $F_C$ yields

$$F_p = F_A \cos \angle ABG + \cos \angle CBG.$$

We proved above that $F_A / F_C = \sin \angle CBG / \sin \angle ABG,$ so we can substitute here and obtain

$$F_p = \frac{\sin \angle CBG \cos \angle ABG + \cos \angle CBG \sin \angle ABG}{\sin \angle ABG}.$$

and multiplying $\cos \angle CBG$ by 1 = $\sin \angle ABG / \sin \angle ABG,$ and rearranging terms in the equation, we get

$$F_p = \frac{\sin \angle CBG \cos \angle ABG + \cos \angle CBG \sin \angle ABG}{\sin \angle ABG}.$$

Now $\angle CBG + \angle ABG = \angle ABC,$ and the Greeks knew that

$$\sin \angle ABC = \sin \angle CBG \cos \angle ABG + \cos \angle CBG \sin \angle ABG,$$

and substituting,

$$\frac{F_p}{F_C} = \frac{\sin \angle ABC}{\sin \angle ABG},$$

or $F_p / F_C = \sin \angle ABC \cdot \sin \angle ABG,$ as stated in point 5. [Ferguson]

3. The force at $B$ is a constant, expressible as the weight of a segment of chain labelled $C,$ of constant length $a$ (not related to point $a$). [Ferguson]

4. If the chain is of uniform density per unit length (call the density $Q$), then the weight of chain $AB$ will be $Qs,$ which increases or decreases as the chain $AB = s$ is lengthened or shortened; the force at $B,$ which Bernoulli sets equal to the weight of a segment $C$ of constant length $a,$ will be $Qa.$ Therefore, the weight of $AB$ is to the force at $B$ as $Qs$ is to $Qa,$ and therefore as $s$ is to $a.$ Bernoulli expresses this ratio of forces in terms of a ratio of lengths, so that it will be in a form solvable by the method of inverse tangents, as is shown below. [Ferguson]

5. Here $y$ is considered an independent variable (instead of $x$). Accordingly, $d^2y = 0$ This aid for the conversion of independent variables is already found in Leibniz manuscripts of 1675.

Bernoulli cannot complete the calculation, because he lacks the logarithmic function. From

$$dy = \frac{adx}{\sqrt{x^2 - a^2}},$$

it follows that

$$y = a \log \left( x + \sqrt{x^2 - a^2} \right) + \text{const}.$$

or, if the axis system is displaced appropriately in the $y$-direction,

$$y = a \log \frac{x + \sqrt{x^2 - a^2}}{a}.$$

From this it follows that

$$x = \frac{a}{2} \left( e^y + e^{-y} \right).$$

The curve is therefore the catenary, which Bernoulli deals with in a later lecture. [Kowalewski]

6. For the equilateral hyperbola $BC,$ $(x + a)^2 - y^2 = a^2$ or $y = \sqrt{2ax + x^2}.$ Therefore $KC = \sqrt{2ax + x^2}.$ If $BA = a$ is the mean proportional between $KC$ and $KD,$ then $KC / a = a / KD$ or $KD = a^2 / \sqrt{2ax + x^2}.$ Then, if the area of rectangle $AG = aAF$ is to equal the area of $HBDKJ,$

$$HBDKJ = \int \frac{a^2 dx}{\sqrt{2ax + x^2}}.$$
then for the curve \( BE \),
\[
y = EK = AF = \int \frac{adx}{\sqrt{2ax + x^2}},
\]
and
\[
dy = \frac{adx}{\sqrt{2ax + x^2}}.
\]

[\text{Ferguson}]

7. For the equilateral hyperbola,
\[
\frac{dx}{y} = \frac{dy}{x} = \frac{xdy - ydx}{x^2 - y^2} = \frac{dx + dy}{x + y},
\]
therefore
\[
\frac{1}{2}(xdy - ydx) = \frac{a^2}{2} d \log(x + y)
\]
and the hyperbolic sector
\[
ABC = \frac{a^2}{2} \log \frac{x + \sqrt{x^2 - a^2}}{a}.
\]
If this be compared with the formula in footnote 5,
\[
y = a \log \frac{x + \sqrt{x^2 - a^2}}{a},
\]
the correctness of Bernoulli’s assertion is seen.

On page 37, col. 1, line 23, \( A \) must be replaced by \( B \).

[\text{Kowalewski}] \[\text{For this translation, the correction has been incorporated into the text.}\]

8. If the area of rectangle \( AG \) is equal to double the area of hyperbolic region \( ABC \), then we have \( aAF = 2 \text{(area } ABC) \)
or \( y = EK = AF \)
\[
= 2(\text{area } ABC) = 2\left( \frac{1}{2} AK \cdot KC - \text{area } BKC \right)
\]
\[
= 2\left[ \frac{1}{2} (x + a) \sqrt{2ax + x^2} - \sqrt{2ax + x^2} dx \right],
\]
and 
\[
dy = \frac{adx}{\sqrt{2ax + x^2}}.
\]

[\text{Ferguson}]

9. The German edition has \( AE \) here, an error for \( BE \). The expression, “the other endpoint \( E \) again will lie on the desired curve,” taken in light of the two preceding paragraphs, indicates that \( BE \) is meant. This is, in fact, the same error that Kowalewski corrected on the previous page of the German edition: The vertex of the catenary had been mistakenly called \( A \) instead of \( B \). See the last paragraph of footnote 7. \[\text{Ferguson}\]

10. “Hence one must seek a certain curve \( BL \)” whose length \( s \) as a function of \( x \), has the differential
\[
ds = \frac{dx \sqrt{2a + x}}{\sqrt{x}}.
\]

Since \( ds^2 = dx^2 + dy^2 \),
\[
dy^2 = ds^2 - dx^2 = \frac{(2a + x)dx^2}{x^2} - dx^2 = \frac{2adx^2}{x^2}
\]
and
\[
dy = \frac{dx \sqrt{2a}}{\sqrt{x}} = \text{the differential of } KL,
\]
that is, the \( y \)-coordinate of \( BL \). So for \( BL \),
\[
y = \sqrt{3ax}.
\]

[\text{Ferguson}]

11. The logarithmic curve is, because of the constancy of the subtangent, characterized by \( y' = y/c \) [that is, \( dy/dx = y/c \)] and is therefore expressed by \( y = be^{cx} \).

Because of its relationship with the logarithmic curve, Leibniz proposed to use the catenary in place of logarithmic tables, especially when travelling, because voluminous tables were inconvenient “to schlepp across land and sea.” \[\text{Kowalewski}\]

12. Definition of the subtangent: For a curve \( y = f(x) \) with a line \( L \) tangent to it at point \( (x', y') \) the subtangent is the segment of the \( x \)-axis between the \( x \)-coordinate of the point of tangency \( (x',o) \) and the intersection of the \( x \)-axis and the tangent line \( L \). \[\text{Ferguson}\]

13. If the length of \( BE = s \) is considered a function of \( x \), then
\[
BE = s = \int ds = \int _0^a \sqrt{dx^2 + dy^2} = \int \frac{dx}{\sqrt{2ax + x^2}} + \frac{dx}{a} + \frac{dx}{x},
\]
which reduces to
\[
\int \frac{(x + a)dx}{\sqrt{2ax + x^2}} = \int \frac{dx}{\sqrt{2ax + x^2}},
\]
which is the \( y \)-coordinate \( AG \) of the equilateral hyperbola \( BG \). \[\text{Ferguson}\]

14. Because
\[
EE = ds = \frac{(x + a)dx}{\sqrt{2ax + x^2}} \quad \text{and} \quad ER = dy = \frac{adx}{\sqrt{2ax + x^2}}.
\]

[\text{Ferguson}]

15. The proposed constructions may be restated in this form: Suppose the catenary \( EBF \), in Figure 9, to be a chain with very fine links. Step 1. At various points along \( EBF \) cut rods that extend vertically from each chosen point on the catenary curve to the horizontal \( EF \). Attach each rod to the link it touches. Now extend the chain as a straight line. Use the free ends of the still-vertical rods to define a line. An area is now enclosed by the straight chain and the line just defined, and can be measured. Step 2. Form any number of other curves whose end-points are \( E \) and \( F \), and whose length is the same as the catenary \( EBF \). Repeat for these curves the procedure described in Step 1.

Bernoulli asserts that the area derived from the catenary is the greatest of all such areas. \[\text{Ferguson} \]