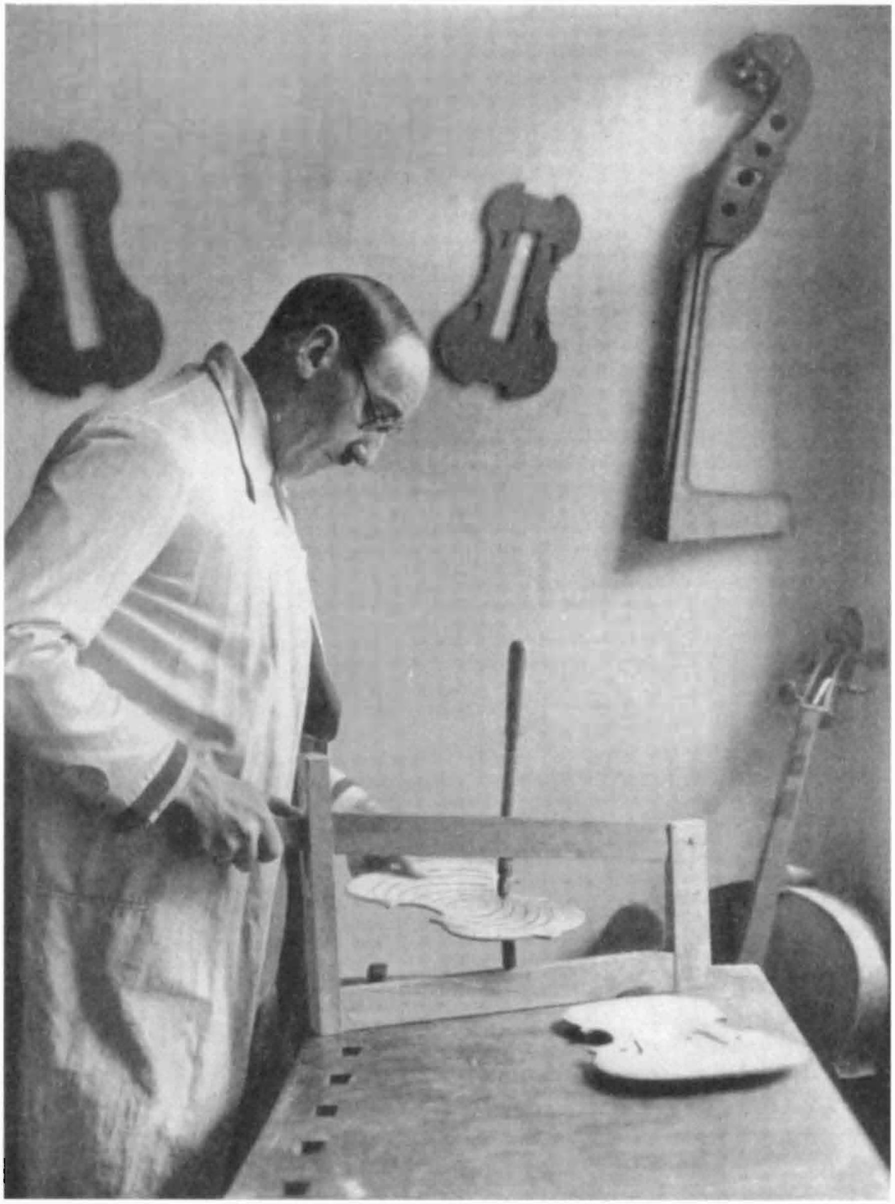
MUSIC AND THE UNSOLVED PARADOX OF PYTHAGORAS

Violin World



Max Moeckel in *Die Kunst der Messung im Geigenbau* (Berlin: Alfred Metzner Verlag, 1935)

The luthier Max Möckel working on the hole-drilling template.

Intervals in classical musical performance cannot be expressed by a predetermined mathematical ratio, nor can the building of great musical instruments be reduced to a formula.

"A beauty of sound, which knows no bounds," is how a luthier once described the tone of the old Italian (and Tyrolese) violins. And, in fact, these old violins embody a concept of instrument-building, which culminated in the great masterpieces of Antonio Stradivari, which can fill great concert halls without any effort, and can be heard over the entire symphony orchestra. Their richness of tone ranges from majestic *fortissimo majestoso,* to the sad lament of an *andante molto grave,* up to the outburst of joy *scherzo prestissimo.* The only other instrument that can be compared to them, is the trained human singing voice. Yet all these tones come from a little wooden box about 35-37 cm long for the violin, 42 cm for the viola, and 74-78 cm for the cello.

When you look a little more closely at these handmade masterpieces, you can recognize that this kind of instrument is a triumph of both acoustics and physics. There are many myths in circulation about these "fabulous instruments." Were the luthiers of old initiated into some secret of the acoustics of physics, which we no longer know today; is there a secret kind of wood, or art of varnishing, which makes all the difference in the sound; or, is there some other kind of secret, relating to how the wood is cut, which had been carefully protected and was lost after the death of Stradivari?

There are still many unanswered questions today about the construction of the violin family (violin, viola, and cello, which all have the same underlying principle of construction), especially after the death of Stradivari, when the great art of creating such wonderful works sank, further and further, into oblivion—and the wildest speculations and most bitter disputes arose. Most of these discussions, however, quite ignore the key question. The Renaissance, the time that saw the birth of this family of

Building and the of Harmonic Sound

by Caroline Hartmann

instruments, was a time of many discoveries, not only in music and instrument-building, but also in painting, the plastic arts, architecture, and machine building. The most significant and many-sided artist and scientist of this time was, without question, Leonardo da Vinci.

The situation in the world of music, and especially regarding musical instruments at the time that we call the Golden Renaissance, provides a number of substantial reasons for assuming that the building of the violin was an invention. The idea for building such an instrument must have sprung from the new discoveries of the Renaissance—above all, from Leonardo da Vinci's investigations into tonality and sound.

However, the Pythagoreans were already acquainted with the paradox, that man recognizes only a few intervals as harmonic, and these intervals, which are heard as harmonic, are created on a stringed instrument by placing the fingers with different spacing between them in each different key. Here, we will pursue this idea somewhat further.

The Unsolved Paradox of Pythagoras

All instruments created by man, use what he has known for thousands of years, that when strings are stretched over a hollow space, more or less beautiful sounds or tones can be created. In India, an instrument of this kind was built around 3000 B.C. Later, Pythagoras (around 500 B.C.), discovered that it was possible to express the relationship between two tones—called intervals—by rational numbers.

Pythagoras invented a one-stringed instrument, a monochord, which the Pythagoreans used for demonstrations, and as a musical instrument. Today, it is used to demonstrate intervals. For example, if you press down on 1/3 of the length of the string, and then pluck or strike it, the resulting tone will be the interval of a fifth above the tone of that same string when it vibrates freely. The significance of his invention was that man recognizes, or experiences, only a few specific intervals as beautiful. These intervals were called *synphon* by the Pythagoreans, and are the following:

EDITOR'S NOTE

We present this piece as a contribution to the pedagogical effort of the LaRouche Youth Movement, which is presently struggling to master the paradox of the Pythagorean comma. Their crucial, related purpose is to attempt to revive some aging intellects of the Baby Boomer generation, who have denied these youth a future by their immoral abandonment of the principle of truth. Readers may find an audio archive of a panel presentation on the Pythagorean Comma by members of the LaRouche Youth Movement, at the website www.theacademy2004.com (click on "2003 Schiller Institute/ICLC President's Day Conference," on the home page).

This article originally appeared in the German-language *Fusion,* (July-Aug.-Sept. 2002). It was translated into English by Richard Sanders. The author, Caroline Hartmann, is an organizer with the LaRouche political movement in Germany. She plays the violin in the Schiller Institute orchestra.

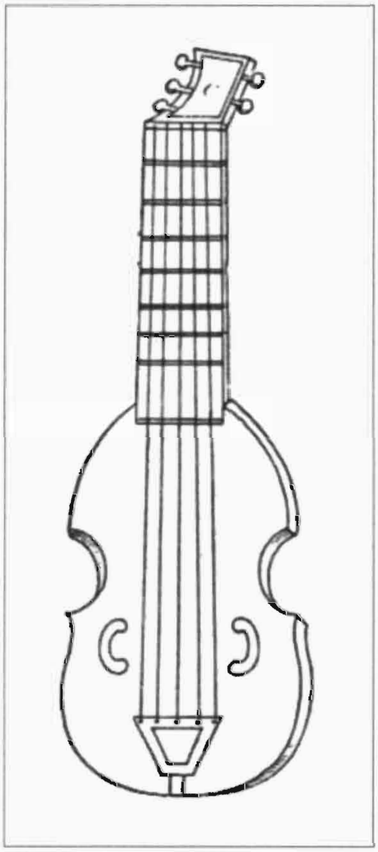
EIRNS/Stuart Lewis

Anna Shavin (left) and Jennifer Kreingold at the LaRouche Youth Movement panel presentation during the President's Day Conference Feb. 16, 2003 in Reston, Va. Kreingold sang a sequence of natural thirds, to demonstrate that they do not close at the octave.

- Octave (ratio 1:2),
- Fifth (ratio 2:3),
- Fourth (ratio 3:4), and
- Third (ratio 4:5).

In addition, there is also the 5:6 ratio, which is the minor third.

The Pythagoreans possessed an 8-stringed lyre and kitharra. All the stringed instruments taken as a whole, up to the beginning of the 16th Century—that is up until the invention of the violin family—had the following characteristics, which significantly limited the quality of their sound, and did not leave much room for expressing a variety of the scale's tone colors (for more on this, see Appendix 1):



Viola da gamba with frets, from Gerle, Musica, 1532.

(1) The fingerboards of these instruments are divided by small ridges, called frets, most familiar to us today from the guitar. The pitch is determined beforehand by these frets, so that for "pure" playing in all the keys one often has to make compromises. Depending on the kind of instrument, there was a certain tempering chosen which allowed for playing in the greatest possible number of keys. One aspect of this, is that the distance from one fret to the next is always different; whence there were naturally many different temperings. When the limits of each instrument's tempering were reached, it had to be retuned, which was the general practice. The discrepancy between the notes sounded on the frets and the proper pitches, as the musician moved through different keys, is sometimes described as the problem of the *Pythagorean comma.*

(2) As for the sound, the resonance chambers of these instruments were for the most part quite flat, or as is the case with fiddles, lutes, or many viols, arched according to certain specific geometrical forms (a cylinder), or with a shape taken from forms in nature. This, from the start, put a limit on the capacity of providing for a "real" or peer-quality accompaniment to the trained bel canto voice. Moreover, the bridge of the instrument is not curved, so that the bow cannot avoid touching all the strings at once, which means that only chords can be played. This kind of limitation can be easily recognized in the accompanying painting of the angel by Fra Angelico (p.19).

The new instrument family of the violin, viola, and cello were revolutionary relative to both these points. The characteristic vaulting curves of these instruments have remained unchanged until today, the instruments showing the same proportions down to the smallest detail. Unlike almost all of man's other inventions, this form has stayed unchanged for 550 years.

Moreover, the paradox of the colors of the tonal scale is solved with genius: They simply eliminated the frets, so that the player himself can determine the pitch and how he will play it. Other than the human singing voice, there is no other instrument which allows this. What a revolutionary breakthrough in music! The instrumentalist could finally "sing" with his instrument, as we know today, from hearing the great violin, viola, or cello *virtuosi.* These two points also prove that there is no way that the violin family could have developed stepwise from some other instrument.

The luthier Max Möckel, who worked around the turn of the 19th Century in St. Petersburg and Berlin, did not rest until he had investigated the true origin of the sonorous and architectonic beauty of the violin. His idea was to investigate whether, in the light of the knowledge of the Renaissance, it might not be possible to discover what part had been played by Leonardo da Vinci, Luca Pacioli, and Albrecht Dürer in the revolution in instrument building. Thus, he began to look for clues to support his hypothesis in the works of these great artists, and he came to the following conclusion:

> Is there really an Italian secret? Yes and no. If we think of it as some kind of recipe, hidden somewhere in some old chest, then no. . . . We must put ourselves into the time in which the violin was invented, and the ideas out of which each of the old masters created their works . . . The most significant minds, to name but two of them, Leonardo da Vinci and his friend Luca Pacioli, had shortly before concerned themselves, in their work of so many facets, with mathematical problems, and when they saw the triangle and the pentagon, they did not see them as merely simple geometrical figures, but they saw in the pentagon, for example, the secret eye of God, a living sensuous image, with its infinite number of unfoldings, for everything that is becoming.

With this hypothesis as a starting point, Möckel developed a procedure for building the violin, viola, and cello, whose standard was what Luca Pacioli called the *Divine Proportion.* (In the Divine Proportion, the division of a line or a geometrical figure is such that the smaller dimension is to the greater as the greater is to the whole.) From that time on, he built many excellent instruments according to this method.

Lawfulness in Nature

As everywhere in Nature, beauty comes from an inner lawfulness. The famous Italian artist Leon Battista Alberti, who was studied in-depth by Albrecht Dürer, once said:

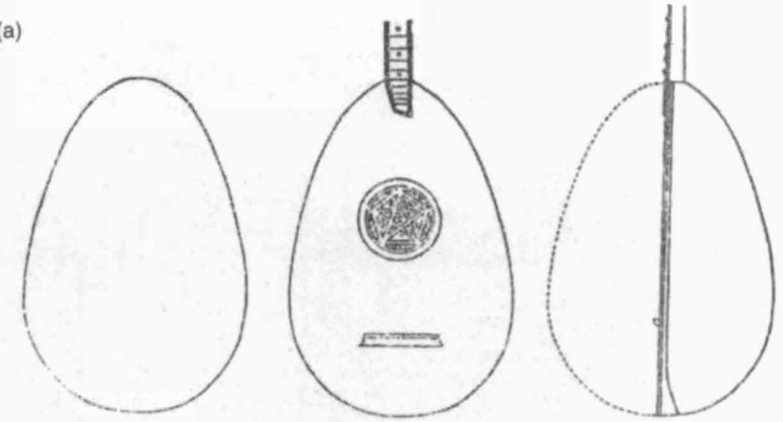
> Beauty is a specific lawful agreement among all the parts, which consists in the fact that you can neither add anything, nor take anything away, nor change anything, without its becoming less satisfying.

Leonardo and his friend Pacioli also knew, that in self-similar *(a) The lute is a perfect egg, from the front and the side view. This curved form is the reason for its tonal possibilities, far exceeding the other instruments.*

(b) An angel painted by Fra Angelico, with fiddle and bow. The bridge of this instrument is not curved. Thus, the bow cannot avoid touching all the strings at once, which means that only chords can be played. Moreover, the fingerboard is divided by frets, and the instrument is flat.

(c) The external form of this Lira da braccio *is similar to the human body. The belly, on the other hand, is practically flat.*

(a)



(b)

(c)

growth processes, you very often find the proportion that Luca Pacioli called "divine" (also called the Golden Section). They also recognized that this was not merely a beautiful principle of mathematical constructions, but the principle of life. Still, the decisive question is: What is the power which creates just this proportion when growth occurs? The luthier Möckel developed a method for adopting this same proportion as the fundamental principle for instrument building.

Nicholas of Cusa, the philosopher-scientist who in his time was in touch with all the great scholars, brought a decisive revolutionary idea along these lines into the scientific discussion: the idea, that all curved lines, such as circles, arcs, and so on, can be expressed through straight lines. In doing so, he created the basis for being able to construct and present curves mathematically or geometrically. Cusa knew what this meant for the further development of music. As he wrote:

> Moreover, from the above, the following is established: just as each straight line can be the side of a triangle, a square, a pentagon, and so on, in the same way, one might find an uncountable number of curved lines, which are like unto a given straight line; therefore one can also find angles, which act like a given straight line, that is, as the side and diagonal in a square, or the radius to the circumference of a circle, and likewise in all the planes, which behave as the given straight lines.
>
> Hence it is possible to come to further conclusions, which have until now been hidden not only to geometry, but were also unknown to music and musical instruments, so that to him who will do his utmost to understand it, there will be disclosed in all its clarity, what was absolutely susceptible of being known in geometry, but was not really known. [Nicholas of Cusa, *Mathematische Schriften*]

Leonardo da Vinci, who, as a painter, most painstakingly studied nature and man, was certainly familiar with these new ideas. But he was not only a painter, but foremost, a researcher into nature, an engineer, architect, make-up artist, sculptor, musician, and much more. Above all, he was interested in the inherent lawfulness of Nature. Following the example of Leonardo, the luthier Max Möckel transposed what the Renaissance had learned about the geometrical construction of the human body, to the construction of the violin. The span from the thumb to the index finger of the left hand served him as the standard length (*mensur*) and point of departure. This distance is the *mensur* of the instrument to be built; namely, the distance from the bridge to the end of the resonance box. Möckel's further geometric construction was based upon two adjacent, upright pentagons, within which hangs a freely float-

The mensur *is the distance between the bridge and the upper edge of the violin body.*

ing square. From there, he developed three small, right-angled triangles which form the basis for constructing all the other details (see Appendix 2).

Furthermore, in all the instruments of the violin family, we observe a multiple curvature, unchanged for 550 years. One such curvature is recognized from the outside, in the form of the arching of the back and the belly of the instrument. The other is only visible to the luthier. It consists of the curvature of the thickness of the wood. Namely, the wood is thicker near the bridge than it is at the sides, towards which it flattens out. And on the sides, there is a narrow strip of wood which has no variation at all in its thickness.

The extraordinary significance of the curvature of the wood can be easily verified by an experiment with wine glasses. If you take a wine glass which starts out thick and flattens out towards the rim, and strike it, you will create a beautiful, powerful, and enduring sound; if on the other hand, you strike a glass whose thickness does not vary, you will get only an unpleasant "noise," or rattle.

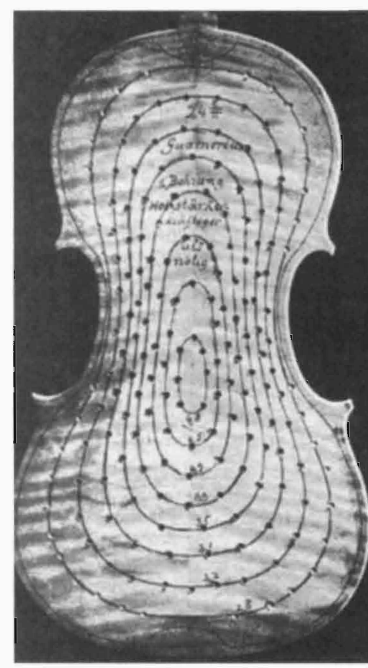
Throughout years of research on many old Italian violins, Möckel ever and again confirmed the proportion of the arched curvature, down to the last detail, from which he derived the idea for his construction. For this he used a method discovered by one of his brothers, Otto Möckel. This was the result of the rediscovery of the so-called "contour circle," found among the tools left behind by Antonio Stradivari, which the old luthiers used to construct regular contour lines, for making the arched curves.

If you were to make a horizontal cut across a mountain range, you would get the contour lines such as those produced by geodetic measurements, which show up on topographical maps.

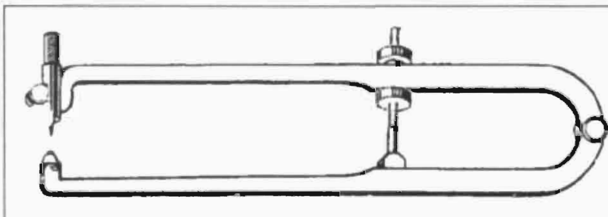
The surfaces of the violin, viola, and cello bodies are irregularly curved everywhere towards their centers, and get more flattened out toward the sides. Of course, the body of the violin is no massive wood plate, in which you could make a horizontal cut, in order to study the interesting arched curves, but it is hollow inside. Two curved surfaces constitute the resonance box of the violin, and it is upon their precise calibration that the quality of the instrument's sound depends. Thus, to study their curvatures, you have to reverse the procedure; you have to take the measure of the vaulting of an instrument previously constructed, and duplicate it by hollowing out a massive wooden board.

Then Otto Möckel "invented" a way of copying the contour lines of all the old instruments. He described his method of working with the peculiar contour circle thus:

> It is to be assumed, that . . . they [the old luthiers] positioned the unfinished parts between the circle openings in such a way, that the pencil point would stand at a perfect right angle against the vaulting, and then [they] moved it lightly. Then, [they] move the circle with a light touch around the vaulting that has not yet been smoothed out, and thus black marks would be made only at a certain height, and the resulting curve drawn will immediately show even to the unschooled eye, all the faults and the wrongly placed contour lines of the vaulting. The mistakes can then easily be rectified, by using a finely adjusted thumb plane to transform the edges and corners of the ugly lines into noble curves. Then the circle, newly adjusted, is brought into action once



The boring of the vaulting curves. ►



Möckel's contour circle tool, which his brother Otto rediscovered among the tools of Antonio Stradivari. The pencil point is at the left.

> again, and the smoothing begins anew. The more curves you draw with the contour lines at different heights, the more the faults will show up.

Thus the construction of the vaulting curves is the basis for the optimal sound distribution over the back and the belly. Luthiers today use Otto Möckel's procedure to copy as precisely as possible the vaulting curves of the old Italian violins.

The Fascinating World of Sound Waves

In order to fully understand the splendid sound of these instruments, however, one must also look at the physics of tonality and sound waves. What is sound, really? What is the origin of the intervals in sound-space, such that we do not recognize all sounds as *synphon,* and what is the source of their creation?

Research into sound clearly confirms the curvature of the old Italian masters as ideal for creating the most powerful, and untrammeled sound, and also for suppressing certain undesirable "wolf tones," or shrill regions of the sound spectrum.

The sound-space itself is "curved" in manifold ways. The many waves of sound which we perceive with our ears as noise or sound, are quite invisible, yet with the aid of experiments, they allow themselves to be visually represented.

We do not know whether or not Leonardo carried out a detailed investigation into the human ear, or hearing per se; but there are many statements in his diaries and drawings, which make one strongly suspect as much. Moreover, there is an outline for experiments, in which he analyzed various kinds of waves, and compared them to one another.

In this research Leonardo compared all kinds of vibrations—light, sound, and magnetism—with one another, in order to discover their similarities. To take this as a starting point, to try to find out what these vibrations might have in common, is still virtually taboo today. Admittedly, in the early 19th Century, André-Marie Ampère recognized that light rays and heat rays were both waves, only distinguished by different wavelengths, and today we know that magnetic and electric rays, X-rays, radio waves, and so on all belong to the same species of electromagnetic radiation. In spite of that, sound waves and water waves have been treated up to now, as if they belonged in a universe with different laws.

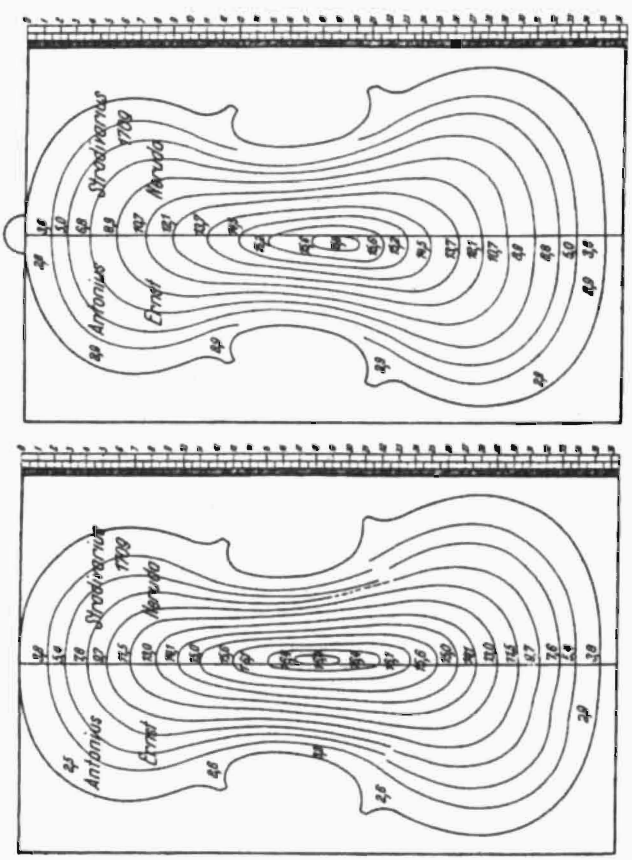
Sound and water waves offer a wide field for investigating the phenomenon of waves, and the lawfulness related thereto. The next scientists to investigate waves of all kinds, after Leonardo da Vinci, were the brothers Ernst Heinrich Weber and Wilhelm Weber, as well as Savart, Poisson, and Benjamin Franklin. In the 1825 work *Experimentally Based Wave Theory, or Concerning Waves of Droplet-forming Fluids with Application to Sound and Light Waves* [Wellenlehre auf Experimente gegrundet, oder ueber die Wellen tropfbarer Flussigkeiten mit anwendung auf die Schall-und Lichtwellen], the Weber brothers defined the concept of wave crest, wave trough, wave amplitude, and wavelength (which they called "wave width"), and they investigated interference phenomena especially closely. You could say that they carried on where Leonardo had already prepared the way.

But what actually happens inside an instrument when sound is created? And how is it that waves are spread? Some active power forces the atoms or the molecules of the medium into motion. In wood, water, the bones of the skull, or the lymph liquids in the ear, or in elastic media, sound waves cause local changes in pressure, and many parts are moved into motion all at once, as if they had received one single blow. The Weber brothers had been inspired by their teacher, Ernst Florens Chladni, to undertake an intensive study of acoustic and musical paradoxes. Chladni himself also had chosen to create—as did Benjamin Franklin—new musical instruments, and for this reason carried out many experiments to try to make visible the motion of waves along a resonance plate. In older physics books, efforts to propagate waves in plates are still described.

In the process of his experiments, Chladni made the following fascinating discovery: Different tones manifest specific characteristics, where some parts move, and some adjacent to them, do not, and it is possible to "see" this. He sprinkled sand onto metal plates, set them vibrating using a violin bow, and then discovered that the sand was formed into various figures on the plates. Depending on the strength of the stimulation, its direction, or its velocity, very interesting sound figures, were formed, demonstrating that sound waves obey specific, peculiar "conditions."

In his works on sound, Wilhelm Weber described Chladni's efforts as follows:

> If you take a round piece of paper, with a diameter of about 8 to 12 inches, and stretch it on a ring—or better, stretch a membrane horizontally over a large glass,



The rediscovery of the function of the contour circles, made it possible to reproduce the vaulting patterns of the old violins. A Stradivarius is diagrammed here.

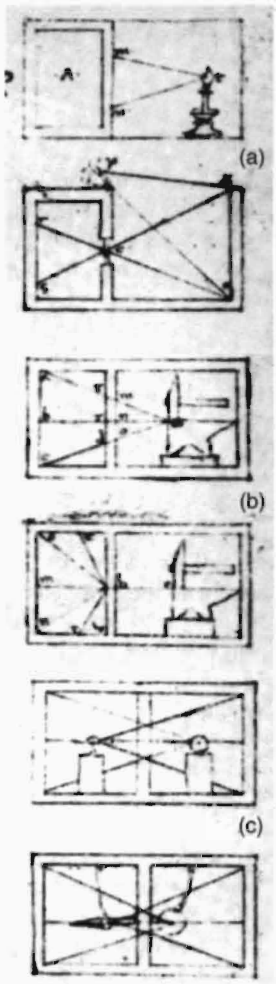


Figure 1
LEONARDO'S SKETCHES FOR AN EXPERIMENTAL SERIES
Leonardo compares the transmission of light (a), sound (b), and magnetism (c). Concerning light, he notes "How the lines, or better rays, of light pass only through transparent bodies." On sound: "How the sound lines penetrate a partition," and "the voice of the echo." On magnetism: "how the lines of the magnet and those of the iron pass through partitions, but the lighter of the two is pulled by the heavier," and "when the weights are equal, both pieces are drawn to each other equally."

which has a stem and base, strew sand on it, and bring a sounding piece of glass near it, about 4 to 8 inches away—then the sand arranges itself into lines, which often form perfectly regular figures. . . . As shown by Chladni, to bring out these kinds of waves, you have to hold the membrane down in several places, for example, at two points of the edge and one on the surface itself . . . the membrane is placed horizontally. . . . Table V., Figs. 6 to 18, present the most regular figures [some of these are shown in Figure 2, below] which are formed upon the moderately vibrating membrane. Where the membrane is not completely taut, then at times a large number of sand lines appear, such as shown in Figure 19, intersecting one another, which seem to originate from the crossing of the circular lines with the diameter lines. [*Wilhelm Weber Werke,* Vol. I, pp. 113-114]

Today, research of this type is going on to try to make visible the wave action on the back and the belly of violins. Laser holograms are also being used to be able to see the vibrations of guitars.

Otto Möckel described the significance of Chladni's research for the understanding of sound-space:

Chladni has now succeeded in making his sound figures visible on freely vibrating round, rectangular, and oval plates. The figures he has discovered, are produced in rich and abundant fullness, when a bowed instrument is played (or, for that matter, any instrument that has a resonance box), and sections of a surface vibrate in continuously changing multiplicity. It is known that each tone, when its vibrations are transferred through a medium to a membrane or to a resonance box, divides the vibrating surface into various parts (depending on its frequency), which do not participate in the movement. . . . Should one draw the vibration waves which are made by various musical instruments, the beauty of these forms would enchant the eyes. All creation of waves—it does not have to be on a musical instrument—contribute to this infinite sea of vibrations. It is not even necessary to have a good imagination to visualize this surging wave painting. We throw a rock into a still pond, and are happy with the circle, that spreads out and surprises us in the uniform round dance of its waves. Now, think of this circle transformed into a sphere, which is continuously increasing in size and intersected by other spheres, without losing their form. In each sphere is a mid-point, the wave-creator, which forms new waves more complex in curvature. That is the soundless matter, in uncanny stillness, which is summoned to allow an invisible miracle of beauty to come into being in rich abundance. These waves are not meant for the eyes, but for the ear. . . . [Otto Möckel, *Die Kunst des Geigenbaus,* pp. 114 and 189]

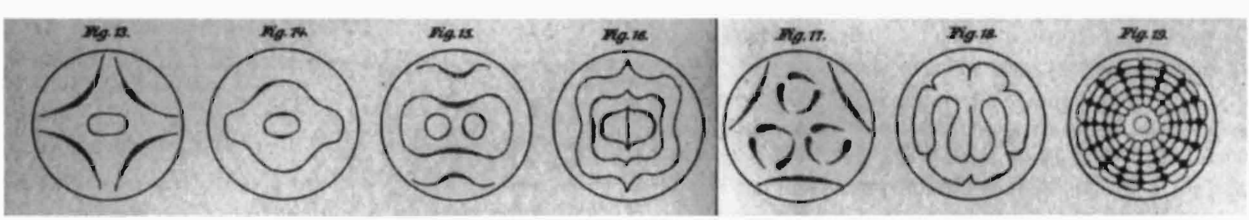
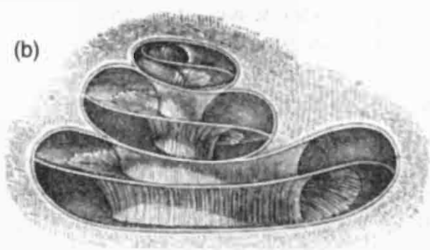


Figure 2
WILHELM WEBER'S SOUND FIGURES
Following Chladni, Wilhelm Weber spread sand on a round piece of paper which he allowed to vibrate. He thus produced characteristic sound-figures, which can be created in the same way on the body of a violin or guitar.

(a)

(b)



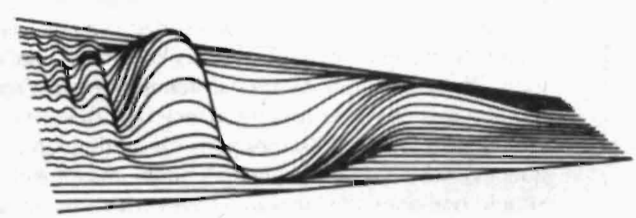
(a) The bony labyrinth in the petrous bone of the skull: at right is the equilibrium apparatus with its three semi-circular canals; on the left, the cochlea (snail) of the inner ear.

(b) Cross-section through the cochlea (snail).

(c) Representation of the basilar membrane of the inner ear stimulated by a sound wave.

The crown of the violin shows the same snail pattern as the cochlea of the inner ear.

(c)



The Human Ear and the Idea of Multiple Curvature

How does the human ear actually hear these phenomenal wave-figures, whose most distinguishable sound-figures the eyes can see? Here a surprising fact comes into play, where once again the relationship which Leonardo and Pacioli so significantly called the *Divine Proportion,* plays a role: namely, that the inner ear is formed like a snail.

In the cochlea (snail), vibrations that strike the eardrum are transmitted by the ear ossicles (hammer, anvil, and stirrup) to the inner ear, and sound vibrations in the air are transformed into signals in the nervous system. This coupling between sound energy and nerve impulses is well known; however, we know very little about the way this variety of information, or impressions upon the nervous system, is processed.

The great significance of the cochlea should also be clear by its being located (together with the labyrinth, the organ that governs our sense of equilibrium), within the petrous bone, the hardest bone in our body. It is a system comprised of three different coiled tubes, placed side by side: the *scala vestibuli,* the *scala media,* and the *scala tympani.* The *scala media* is filled with a fluid called endolymph, which is high in potassium and low in sodium, in contradistinction to the perilymph present in the *scala vestibuli* and *scala tympani,* which has exactly the opposite composition. This difference creates an electrical potential between the two liquids. Between the *scala media* and the *scala tympani,* there is a fibrous membrane, called the basilar membrane, on which the hair cells are to be found. These cells act as very sensitive sensors, transmitting the sound waves entering from without, in such a way that the membrane begins to vibrate in three-dimensions. High notes tend to stimulate primarily the hair cells at the opening of the basilar membrane, low tones stimulate the hair cells at the other end. The electric currents that arise are transmitted by nerve fibers connected to the hair cells, along the auditory nerve and to the brain.

It is more than symbolic that Man's auditory organ should be in the spiral form of a snail shell, given the lawful connection between music and geometry. What other explanation could there be for the snail's playing such a significant role as the head of the violin, from the beginning of the 16th century until today? Why has it become practically a standard form—as a three-dimensional ornament, going from the peg box, to a sculpted scroll which gets steadily wider, so that its greatest breadth is reached at the midpoint of the snail? After all is said and done, the snail forms the crown of the new family of instruments, the violin, viola, and cello; perhaps it strengthens the waves or acts as a wave-guide; one thing is certain: it expresses the inner lawfulness of the construction of the instrument.

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APPENDIX 1

ON THE CAUSES OF THE DIFFERENT COLORS OF THE SCALE

'A Completely Pure Music Is Absolutely Impossible'

The researcher Ernst Florens Chladni (1756-1827), teacher of the famous physicist Wilhelm Weber, explained why "pure" music is not possible:

> In whatever kind of *tempering* you might choose, a small deviation from the general purity of the tonal relationship, is *indispensable*—not only for instruments with a fixed pitch, as many believe, but in general. The reason for this lies in the nature of the number relationships themselves, such that, if you want to assert each tone purely against the fundamental, they will not have a pure relationship to each other, and if you want to give to each tone a pure relationship to the one preceding and the one following, then the relationship to the fundamental is lost. A completely *pure music* (that is, where each fifth is equal to 2:3, each major third is equal to 4:5, each minor third is equal to 5:6, and so on) is thus absolutely impossible, even if one were to stay within one diatonic key, but also if you wish to distinguish all the raised and lowered intervals precisely. [*Kurze Uebersicht der Schall und Klanglehre* (Brief Overview of the Science of Sound and Tonality), p. 12]

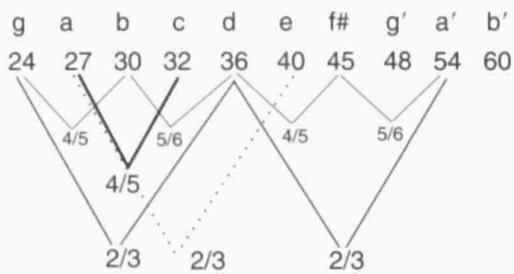
The author with her violin.

In this context, look at the following example: If one presents the individual steps of the scale, *g, a, b, c, d, e, f#, g'*, and *a'* (*a'* is added here for purposes of demonstration) as frequencies with the smallest whole numbers, 24, 27, 30, 32, 36, 40, 45, 48, and 54 Hz, immediately one comes up against the following difficulty: The frequencies are so chosen, that *g-b-d*, as well as *c-e-g'*, and *d-f#-a'* form a major triad, that is, the first interval of these triads (*g-b, c-e*, and *d-f#*) is in the ratio of a major third (4/5), and the intervals *b-d, e-g'*, and *f#-a'* are in the ratio of a minor third (5/6) to each other. Each of the cited triads gives the ratio of 2/3, thus a fifth. But it is not possible to form a fifth on the tone *d* within the same scale.

If, using that same *d* as a starting point, you would use the same tones to form the *d*-minor or *d*-major scale, the result would sound quite impure to the ear. The relationship *a/c* = 27/32 is neither a major nor a minor third (that is, neither 4/5 nor 5/6); nor is the ratio *a/e* = 27/40 a pure fifth, but slightly larger. The ratio of a pure fifth to *d*, should be 27:40.5. But that is too low for the *d*-scale (minor or major).

In musical praxis, it is shown that the ear (or the human spirit) knows very well how to deal with this paradox, and even to utilize it *creatively*. It continuously, and *as much as possible*, smoothes out the little "impurities" or "ambiguities," which arise from the above-cited phenomenon of the difference in keys. This is difficult to do with instruments whose pitch is fixed, but on the violin, viola, or cello it is very easy. The "fitting," or "tempering," means that for the purpose of overall musical harmony, the ear abandons the absolute arithmetical "evenness" of the same tones. The decisive reason for this was *discovered by Leonardo da Vinci*. *The* ear hears the music not as single notes; rather the music is apprehended as intervals, not as notes. Leonardo wrote:

> Each impression endures momentarily in the object receptive to it; but the stronger impression will be held longer, and the weaker, shorter. I call *receptive*, in this case, any object which is changed from its original situation by any kind of impression whatsoever, and *unreceptive* that object, which might indeed have been changed from its original situation, but it fails to retain any impression of what it was that moved it. Receptive is the condition when a blow is impressed upon a sounding object, such as when bells and the like are struck, or sound in the ear. Were this impression of tones not to endure, then a one-voiced song would not have a beautiful sound: for when you go quickly from the first note to the fifth, it is as if you hear both tones as the same time, and therefore, you hear the harmony which the first makes together with the fifth. But if the impression of the first were not to remain for some time in the ear, then the fifth, which follows right after the first, would sound alone. And since one tone alone cannot form a harmony, such a one-voiced song would not be beautiful. [Leonardo da Vinci, *Tagebücher und*



g/d = 24/36 = 2/3 (pure fifth)

d/a′ = 36/54 = 2/3 pure fifth

but:

a/c = 27/32 ≠ 5/6 (not a minor third)

a/e = 27/40 ≠ 2/3 (larger than a pure fifth)

THE TEMPERING PARADOX

Here is the scale of g, assigning to the frequencies the least possible whole numbers, which allow for the formation of the harmonic intervals—the fifth, fourth, minor and major third—in relation to the fundamental frequency.

Aufzeichnungen (Notebooks), p. 390]

The great task when building instruments, still today, is to accomplish the most "precise" possible tempering, which comes the closest to the human ear. The "working process" of the human ear must become reconstructible, in order to make the most beautiful music possible. Before the invention of the violin, viola, and cello, that was not so easily done, because the notes of plucked or bowed instruments were predetermined by the frets. Thus, each time musicians playing well-tempered music wanted to shift keys, they had to retune their instruments.

APPENDIX 2
DIFFERENT COLORS OF THE SCALE

The Construction of The Violin's Vaulting Curves

The luthier Max Möckel based his construction of the vaulting curves, as well as the gradient of the thickness of the wood of the violin's back and belly, on the Golden Section. Here is his construction for the gradient of the thickness of the wood.

Let us consider the square *PQXY* (Figure 2), within which the fundamental form of the violin is enclosed, consisting of three small rectangles. The perpendicular bisector of this square is exactly the same length as the air-space of the violin's body. Let us call this perpendicular bisector *ML*, and the horizontal line perpendicular to it, *NO*. Thus, two rectangles are formed, *PQON* and *NOYX*. Now we must divide the vertical lines *PX* and *QY* according to the Golden Section. The result is a large segment *PK*, and a small segment *KX*, as well as the large segment *XI* and the small *PI*. Similarly, we divide the side *QY* twice into the proportion of the Golden Section: *QH* is the larger and *HY* the smaller part of one division; for the second, *YG* is the larger, and *GQ* the smaller. This process is necessary because we have to find the center of each circle, which will provide us with the curves for the gradient of the thickness of the wood for the back and the belly of the instrument.

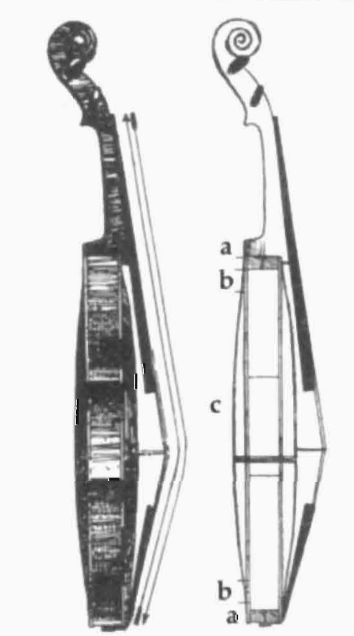


Figure 1
SIDE VIEW OF THE VIOLIN

Let us now connect the points *K* and *H* with *M*, and the points *G* and *I* with *L*; through which intersections the points marked *3* and *4* are obtained. Now, if we draw within the upper rectangle *PQNO*, the diagonals *NQ* and *OP*, we get the point of intersection marked *1*. The diagonal connection between point *3* and *Y* (the corner of the square), and of point *4* with *X*, then gives us the intersection point *2*. Points *1* and *2* are centers of circles crucial for the entire further construction.

As evident from Figure 3, *KM* and *NQ* intersect at point *5*, and the lines *OP* and *MH* at point *6*. Now, if we connect point *7* to *K*, point *8* to *H*, point *5* to *9* and *6* to *10*, we obtain intersections *C* and *D*, which, likewise, will be centers of important circles. We still have to discover the radii of these four circles.

To that end, let us first draw a horizontal line between *C* and *D*, which cuts the perpendicular bisector at point *A*. The line that connects *A* to point *1* is then the radius of the circle whose center is *1*. The horizontal line *HK* cuts the two vertical lines of the middle rectangle at points *11* and *12*, which we connect diagonally with the corners of the square *P* and *Q*. These diagonals cut the perpendicular at point *B*. The connection between *B* and point *2* is the radius of the circle whose center is at *2*.

Now, if you drop perpendiculars from points *C* and *D* to the horizontal line, HK, you will get the radius of the circles around *C* and *D*. With proper construction, these circles will be tangent to the other circles around *1* and *2*.

Now, let us construct the curves of the thickness gradient. If we extend the horizontal line *CD* on both sides until it intersects *PX* and *QY* (Figure 4), then the straight lines *VE* and *WF* will be radii of the circles around *E* and *F*. These circles show some overlapping with the preceding ones. The gray shaded area is the outline of the resonance plate to be worked on later.

The resonance plates are then cut out according to the outline produced in this way. What next? How would they look in cross-section? Max Möckel investigated countless old masterpiece violins, and noted the following: First, all the instruments are "flat" near the edge, where there is no curvature of

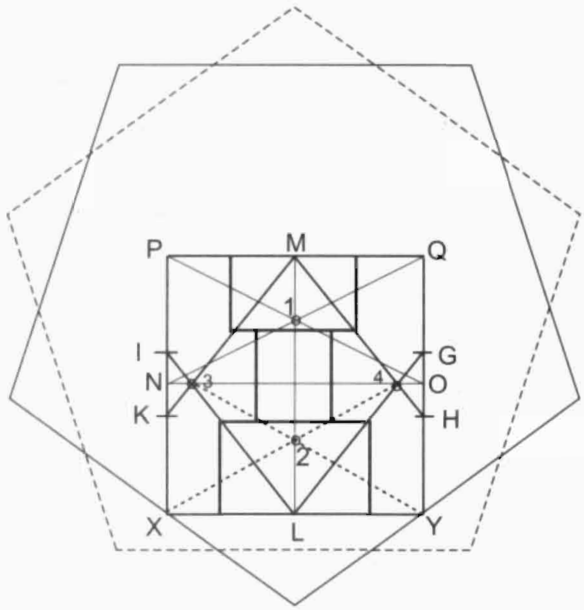


Figure 2
GOLDEN SECTION CONSTRUCTION
The first step in the construction of the wood-thickness gradient within the initial pentagon. The relationship of the diagonal of a pentagon to its side is in the golden section or golden proportion.

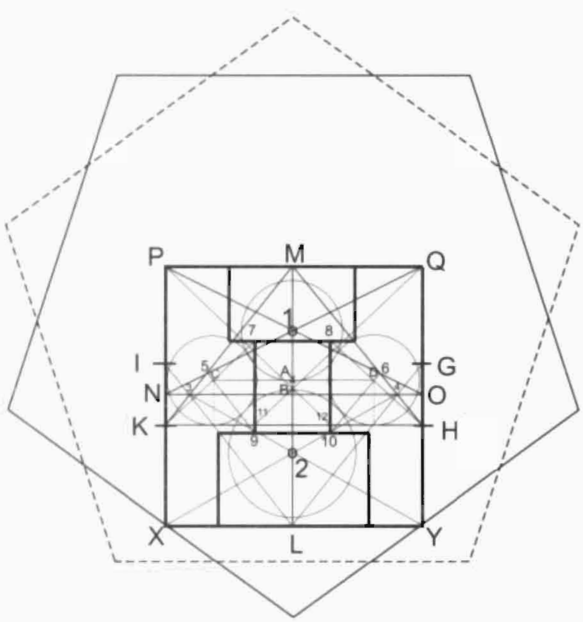


Figure 3
DETERMINING THE PROPORTIONS OF THE VIOLIN
The first four auxiliary circles are determined.

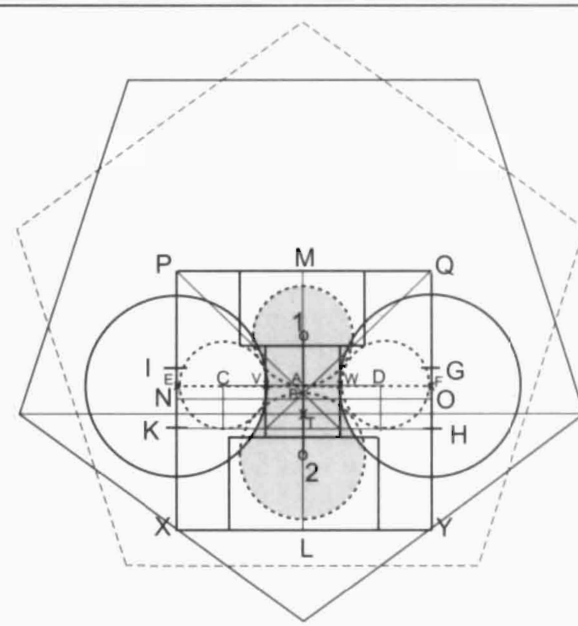


Figure 4
DETERMINING THE RESONANCE PLATES
The four circles, which determine the domain of the resonance plates whose wood-thickness gradient is to be constructed.

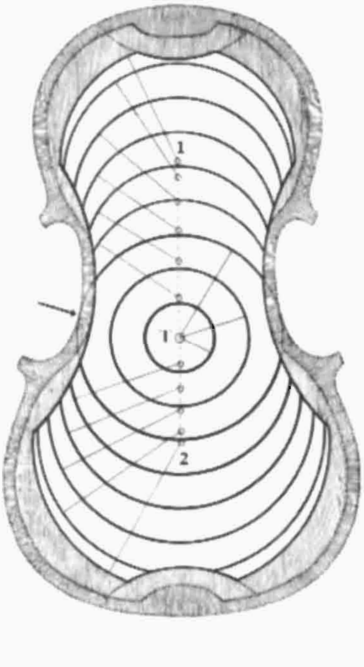


Figure 5
DETERMINING THE WOOD-THICKNESS GRADIENT
The more tangent circles we draw within the outline (determined by the four circles), the more precise will be the wood-thickness gradient.

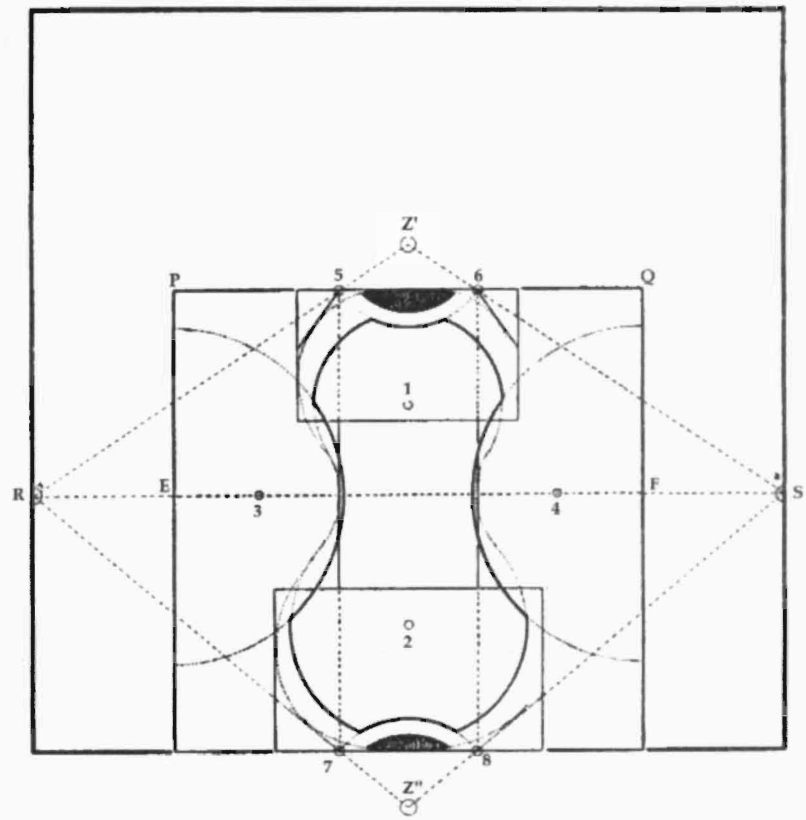


Figure 6
DETERMINING THE NON-CURVED REGION
The region around the edges, which has neither curvature nor a thickness gradient, is constructed. (This is the region marked a in Figures 1 and 7.)

either the back or belly, nor any gradient in the thickness of the wood (Figures 1 and 5).

Second, the maximum thickness of the back and the belly are related to each other according to the Golden Section proportion. Möckel remarked about this:

> It is a fundamental rule, that the mutual relationship of the thickness of the back and the belly, must always be set. This is how the difference between the vaultings of the back and the belly is always tempered in an ideal manner. It is in the relationship, of the thicknesses of the one to the other, where the main secret of the classical Italian masters lies, to my knowledge, in the determination of the relationship of the thickness of the one to the other, and this is, moreover, a splendid explanation for the many different gradients of wood-thickness, which we find in the old Italian masterworks. [Max Möckel, *Kunst der Messung,* (Secrets of Construction), p. 98 ff.]

How is it possible to bring about a lawful gradient in the thickness of the wood? To that end, let us consider the region shaded in gray in Figure 4. Let us mark in the diagonal which runs across the lower part of the pentagon, and call it the *measure line.* The intersection of this diagonal with the perpendicular bisector, we call *T.* Now, we draw three concentric circles around point *T* (Figure 5). The diameter of the smallest circle must be equal to the width of the bridge. (Möckel has explained the reason for this elsewhere; here we will just assume it.) The largest circle is the tangent circle to the outer edge (Indicated by the arrow). The middle circle around *T* can be drawn arbitrarily, somewhere between the largest and the smallest circle.

Using as centers any arbitrary points, as long as they lie between points *1* and *2* and the perpendicular bisector, draw circles which are tangent to the outer edge of the violin. The more such circles we actually draw, the smoother the gradient of the wood thickness, which is the greatest at *T,* and decreases going outward. The circles around points *1* and *2* give the upper and lower limits of the white zone in Figure 5.

Lastly, we still have to determine what to do with the belly and the back areas, how in fact they are to be vaulted relative to their outer form, as well as to their inner thickness gradients. To that end, we extend the horizontal line of the small square *PQXY* out to the edges of the large square (Figure 6), and connect the end points *R* and *S* with points *5* and *6,* and *7* and *8,* respectively.

The intersection of these diagonals above and below the body of the violin, at Z' and Z'', are, in turn, centers of circles with the radii $Z'5$ (= $Z'6$), or $Z''7$ (= $Z''8$). These circles cut a small arc from the shaded inner surface at the top and the bottom. By using the circles that we constructed previously around *E* and *F*, which had already outlined a part of the basic form, we now have created three different surfaces upon each plate of the body of the violin (Figure 7).

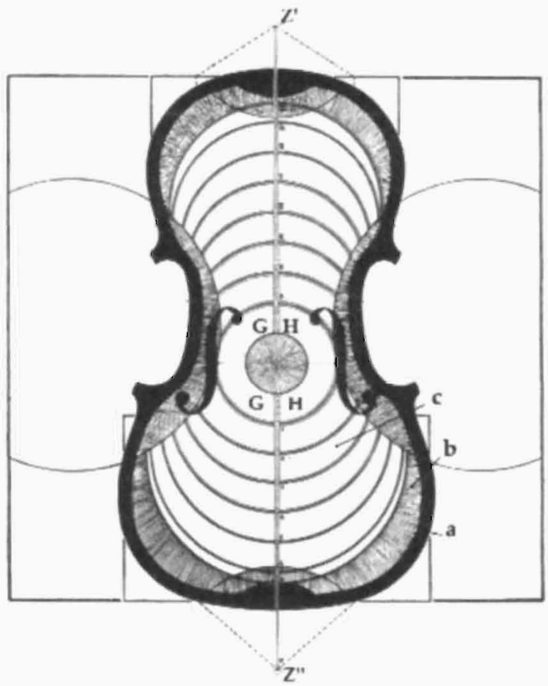


Figure 7
THE THREE REGIONS OF THE RESONANCE PLATE
All three regions of the resonance plate, a, b, and c. (These are shown in side view in Figure 1.)

- The black portion shows the edge of the back and the belly, which remains flat. The ribs (as the surfaces that make up the side of the violin are called) and the corner blocks are glued onto these.
- The gray cross-hatching marks the surface between the edge and the inner part, where the thickness of the wood is not precisely determined prior to the actual finishing, but blends in naturally.
- The white inner part is divided by circular curves, and shows the specific surfaces to be worked on, whose thickness is shown by the points marked.

Using this method, Möckel determined the maximum thickness of the wood for the belly and the back. The straight line *GH* is the thickness of the belly. This is indicated above and below the small circle in the middle. Now, when we connect *G* and *H* above with Z', and connect *G* and *H* below with Z'', we get two elongated, narrow wedges. They are different from each other, and show the gradient of the thickness of the wood of the belly. You can see that the wedge which runs a longer distance to the top of the instrument, is slightly narrower than the lower, shorter one; this indicates the different thickness gradients (Figure 7).

Möckel said the following about this:

> The difference might be small, but it is once again a proof that it is impossible to copy the thickness of the wood without really knowing the construction. The failure, when it comes to tonal quality, of the so-called exact copies, derives most of all, from this fact. [Max Möckel, *Kunst der Messung,* (Secrets of Construction), p. 59]